MATHEMATICAL MODELING OF CRITICAL SITUATIONS IN ENVIRONMENTAL PROBLEMS

V. Vavilin, A. Molchanov

Today, owing to existence of the quick-acting electronic computers and to availability of the sufficiently convenient languages for programming, there are more opportunities to construct the complex models with a great number of variables and parameters reflecting the numerous internal and external links present in real systems.

Nevertheless, every attempt to take into account even some of all properties of a real object must inevitably lead to creation of multidimensional models. Even quite negligible changes of parameter magnitudes may grow to a cause of drastic reconstruction of the regime of such a model. At the same time, however, the numeral parameter values are being derived from data coming from observation and in many instances they are known by the order of magnitudes only.

Ecology plays a leading role in the environmental sciences. There are four specific peculiarities of biological systems which are of importance for modeling :

1/ complexity of internal structure of each individual or, to put it mathematically, multidimensionality of phase space of a biological system,

2/ polyfactoral properties of environment, i.e. conditions of vital activity, demanding a great number of para-

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meters, both continuous and discrete, which cause that a complex environment of the functioning system must be studied,

3/ uncloseness in energetical sense as well as in structural or informational ones, meaning the necessity of "co-modeling" of both biological system and environment of vital activity,

4/ essential nonlinearity, meaning that a great range of external characteristics provides for keeping the viability of systems and thus resulting in essential nonlinearity of the mathematical models.

In view of the above an accurate enough model of the ecological system becomes extremely complicated and the volume of calculation work on such a model is very big. So, the only remaining solution consists in building of the well worked - out approximate models.

Division of biological systems into levels of living, i.e. macromolecules - organnels - cells - tissue - individuals - populations - ecosystems, and, finally, biosphere - makes the task of mathematical modeling much easier. The possibility to ignore the adjacent, i.e. upward and downward, levels is of common character and is closely connected with the idea of a small scale parameter. The environment can be considered as one almost constant as it changes rather slowly within time and space inherent for the object subjected to a study.

The internal space of the object can also be considered as constant or, to say more accurately, dependent only upon the essential variables which describe the object forming a subject of studies. In this case there are only the mean values of rapid variables that count. All that has been said above can be expressed with the use of systems of equations given below :

$$\frac{dx}{dt} = \delta A (x,y,Z)$$

$$\frac{dy}{dt} = B (x, y, Z) \tag{1}$$

$$\varepsilon \frac{dZ}{dt} = C (x, y, Z)$$

where :

x - variables of the external environment /I level/,

y - essential internal variables /II level/,

Z - rapid internal variables /III level/

ε, δ - small scale parameters.

Thus, the general scheme is a three-level model. For a single-level model it is necessary to know the limits of applicability beyond which we may observe the buckling of the system being studied.

In such critical regimes the number of key variables is not too high /usually 2 to 3 / and it allows to predict all the possible kinds of dynamics. The analogs of these kinds are flexible or stiff regimes of stimulation of autofluctuation in radiotechnics, explosion, monomolecular dissociation. A combination of such regimes is also possible.

Similar express-models /2 to 3 essential variables/ allow to realize the essence of the occuring phenomenon, to find characteristic parameters and, what is more important, to determine the critical values of these parameters.

In conclusion we present an example of a two-level mathematical model of biochemical oxidation process of a substance immersed in a basin.

The Streeter-Phelps equation based on two variables, i.e. the concentration of organic substance and of the soluble in water oxygen is a classical model of oxidation of organic matter occurring in water.

In the case of organic matter the above equation simply represents the monomolecular dissociation. However, one of the obvious deficiencies of that model consists in neglecting of influence of bacteria number being fed on the substratum.
To be more exact, this number is considered to be one invariable and is regarded as an environmental factor. When oxygen is in plenty it becomes necessary to introduce the second variable, namely the number of bacteria. It is exactly the reason why one of the possible models of biochemical oxidation is to be expressed in the form of the following set of differential equations :

$$\begin{cases} \frac{dB}{dt} = a\left(1 - e^{-s/b}\right) B - cB \\ \frac{dS}{dt} = d - g\left(1 - e^{-s/b}\right) B \end{cases} \qquad (2)$$

where :

B - number of bacteria per volume unit,

S - concentration of substrata and initial conditions are

$$B_o = B_{stat.} = \frac{ad}{gc}, \quad S_o > S_{stat.} = b \ln \frac{a}{a-c}$$

It is supposed that there is a background concentration of bacteria and the organic matter in basin is a cause of natural rotation.

Suppose, $s/b \ll 1$; in this case, having introduced the new variables $x = \frac{g}{b} B$, $y = \sqrt{\frac{a}{bd}} S$, $\tau = \sqrt{\frac{ad}{b}} t$ we shall obtain the system similar to Lotka-system which has been proposed by Lotka for description of the fluctuable chemical processes :

$$\begin{cases} \frac{dx}{d\tau} = xy - x \\ \frac{dy}{d\tau} = 1 - xy \end{cases} \qquad (3)$$

where : $\alpha = c\sqrt{b/ad}$ and initial conditions are :

$$x_0 = x_{stat.} = \frac{1}{\alpha} \quad , \quad y_0 > y_{stat.} = \alpha$$

While considering the case $\alpha \ll y_0 \ll 1$, for $\tau \ll \alpha \ln \frac{y_0}{\alpha} - 1$ we shall obtain the number of bacteria (x) , i.e. the slow variable of external enviroment and for the organic matter (y) : : $\frac{dy}{d\tau} \simeq x_0 y$, i.e. the monomolecular dissociation.

This is the classical case expressed by Streeter-Phelps. It is easy to show that the investigated stationary state is a stable node when $\alpha < \frac{1}{4}$ and a stable focus when $\alpha > \frac{1}{4}$; that means that the critical value of parameter is $\alpha = \frac{1}{4}$. When $\alpha > \frac{1}{4}$ the system returns to the initial state through the fading fluctuable process. The above example indicates that the concept of environment can be rather conditional. Thus, the introduction of different variables into environment should have the consequent limits of application.